Government of **Western Australia**
School Curriculum and Standards Authority

MATHEMATICS METHODS

Calculator-assumed

ATAR course examination 2020

Marking key

Marking keys are an explicit statement about what the examining panel expect of candidates when they respond to particular examination items. They help ensure a consistent interpretation of the criteria that guide the awarding of marks.

Section Two: Calculator-assumed **65% (97 Marks)**

Question 8 **(7 marks)**

The weight, X, of chicken eggs from a farm is normally distributed with mean 60 g and standard deviation 5 g. Eggs with a weight of more than 67 g are classed as 'jumbo'.

(a) What proportion of eggs from the farm are 'jumbo'? (2 marks)

Solution
$P(X > 67) = 0.08076$
Specific behaviours
✓ states the correct expression for the probability ✓ calculates the probability

(b) What proportion of 'jumbo' eggs are less than 75 g in weight? (3 marks)

Solution
$P(X < 75 \mid X > 67) = \frac{P(67 < X < 75)}{P(X > 67)} = \frac{0.0794}{0.0808} = 0.9832$
Specific behaviours
✓ writes a conditional probability statement ✓ recognises the restricted domain for 'jumbo' eggs ✓ calculates the probability

(c) The heaviest 0.05% of eggs fetch a higher price. What is the minimum weight of these eggs? (2 marks)

Solution
$P(X > m) = 0.0005$ $m = 76.45$ g
Specific behaviours
✓ writes the correct expression $P(X > m) = 0.0005$ ✓ calculates the minimum weight

Question 9 **(8 marks)**

A cake shop makes birthday cakes. The probability distribution of the number of birthday cakes sold in a day, X, is given below.

x	0	1	2	3	4
$P(X = x)$	0.1	0.2	0.25	0.35	0.1

(a) Calculate the mean number or birthday cakes sold in a day. (1 mark)

Solution
$E(X) = 0 \times 0.1 + 1 \times 0.2 + 2 \times 0.25 + 3 \times 0.35 + 4 \times 0.1 = 2.15$
Specific behaviours
✓ evaluates the mean correctly

(b) On Monday, the cake shop makes four birthday cakes. If each birthday cake costs \$20 to make and sells for \$50, what is the expected profit or loss on that day? (3 marks)

Solution
The cost of making four birthday cakes is $4 \times \$20 = \$80$ The expected number sold is $E(X) = 2.15$, so the expected income is $2.15 \times \$50 = \$107.50$ Thus, the expected profit for the shop is $\$107.50 - \$80 = \$27.50$
Specific behaviours
✓ calculates the cost of making the birthday cakes ✓ uses the expected number sold to calculate the expected income ✓ calculates the expected profit

On Tuesday, the shop makes three birthday cakes. Let the random variable Y denote the number of birthday cakes **not** sold on that day.

(c) Explain why $P(Y = 0) = 0.45$. (2 marks)

Solution
$Y = 0$ when all the birthday cakes are sold So, the number of birthday cakes requested for sale is 3 or 4 So $P(Y = 0) = P(X = 3) + P(X = 4) = 0.35 + 0.1 = 0.45$
Specific behaviours
✓ relates the value of Y to the values of X ✓ obtains the probability as a sum of the two probabilities

(d) Obtain the probability distribution of Y. (2 marks)

Solution

y	0	1	2	3
$P(Y = y)$	0.45	0.25	0.2	0.1

Specific behaviours

✓ obtains $P(Y = 1)$ or $P(Y = 2)$ or $P(Y = 3)$
✓ completes table correctly

Question 10 **(7 marks)**

Water flows into a bowl at a constant rate. The water level, h, measured in centimetres, increases at a rate given by

$$h'(t) = \frac{4t+1}{2t^2+t+1}$$

where the time t is measured in seconds.

(a) Determine the rate that the water level is rising when $t = 2$ seconds. (1 mark)

Solution
$h'(2) = \dfrac{4(2)+1}{2(2)^2+(2)+1}$ $= \dfrac{9}{11}$ cm/s $\{0.818\}$
Specific behaviours
✓ determines correct rate including units

(b) Explain why $h(t) = \ln(2t^2 + t + 1) + c$. (2 marks)

Solution
$h'(t)$ is of the form $\dfrac{f'(x)}{f(x)}$ (the numerator is the derivative of the denominator), so the function $h(t)$ is the natural logarithm of the denominator.   Also, +c needs to be included in the function, as any constant could be included here.
Specific behaviours
✓ states that the numerator is the derivative of the denominator   ✓ identifies the number c as the constant of integration

(c) Determine the total change in the water level over the first 2 seconds. (1 mark)

Solution
$\Delta h = \displaystyle\int_0^2 \frac{4t+1}{2t^2+t+1}\,dt$ $= \ln(11)$ cm $\{2.398\}$
Specific behaviours
✓ determines total change

The bowl is filled when the water level reaches $\ln(56)$ cm.

(d) If the bowl is initially empty, determine how long it takes for the bowl to be filled. (3 marks)

Solution
Let the time taken for the bowl to be filled = a seconds $\ln(56) = \int_0^a \frac{4t+1}{2t^2+t+1}\,dt$ $= \left[\ln\left(2t^2+t+1\right)\right]_0^a$ $= \ln\left(2a^2+a+1\right)$ $56 = 2a^2+a+1$ $a = 5$ The bowl will take 5 seconds to completely fill.
Specific behaviours
✓ states a definite integral for depth of water ✓ equates definite integral to maximum water level ✓ determines time taken

Question 11 **(9 marks)**

The line $y = x + c$ is tangent to the graph of $f(x) = e^x$.

(a) Obtain the coordinates of the point of intersection of the tangent with the graph of $f(x)$. (2 marks)

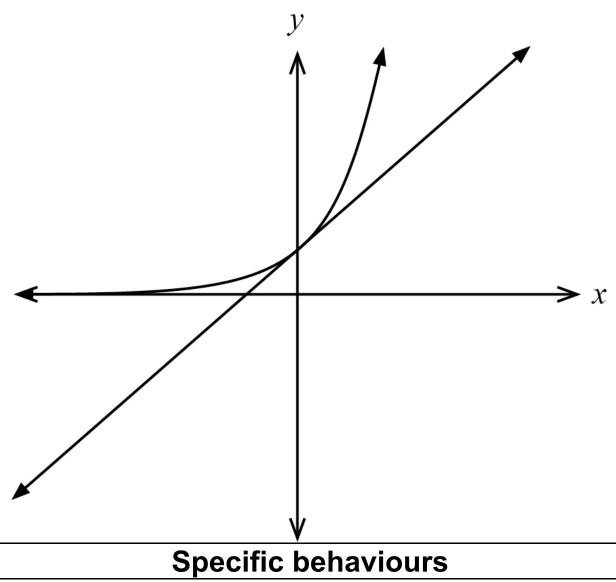
Solution
$f'(x) = e^x = 1$ So $x = 0, f(0) = 1$
Specific behaviours
✓ obtains equation to solve for the x coordinate ✓ states the coordinates of the point

(b) What is the value of c? (1 mark)

Solution
The point $(0,1)$ lies on the line, so $1 = 0 + c \Rightarrow c = 1$
Specific behaviours
✓ obtains the correct value of c

(c) Sketch the graph of $f(x)$ and the tangent on the axes below. (1 mark)

Solution



Specific behaviours

✓ sketches both functions showing tangent at (0,1) with shapes correct

(d) Evaluate the exact area between the graph of $f(x)$, the tangent line, and the line $x = \ln 2$. (3 marks)

Solution
$$Area = \int_0^{\ln 2} (e^x - (x+1))dx$$ $$= e^x - \frac{x^2}{2} - x \Big\vert_0^{\ln 2}$$ $$= 2 - \frac{(\ln 2)^2}{2} - \ln 2 - 1$$ $$= 1 - \frac{(\ln 2)^2}{2} - \ln 2$$
Specific behaviours
✓ writes down the correct integrand ✓ gives the correct limits for the integral ✓ evaluates correctly

(e) Given that $g(x)$ is the inverse function of $f(x)$, write a definite integral that could be used to determine the area between the graph of $g(x)$, the x-axis, and the line $x = \ln 2$. (2 marks)

Solution
$$Area = -\int_{\ln 2}^{1} \ln(x)dx$$
Specific behaviours
✓ recognises the inverse function of $f(x)$ ✓ states the correct definite integral

Question 12 **(21 marks)**

It is estimated that 20% of small businesses fail in the first year. A business advisory group takes a random sample of 500 new businesses that started in January 2018. An analyst employed by the group suggests the use of the binomial distribution is appropriate in this case.

(a) What is the probability that at most 120 of the businesses fail in the first year? (2 marks)

Solution
Let the random variable X denote the number of new businesses that fail out of the 500. Then $X \sim Bin\,(500, 0.2)$. $P(X \leq 120) = 0.9877$
Specific behaviours
✓ states the parameters of the Binomial distribution of an appropriate random variable ✓ calculates the probability

(b) What is the approximate distribution of the sample proportion of small businesses that fail by the end of the year in this sample? Justify your answer. (3 marks)

Solution
Sample proportion $\hat{p} = N\left(0.2, \frac{0.2 \times 0.8}{500}\right)$, That is, $\hat{p} = N(0.2, 0.00032)$, as the sample size is large
Specific behaviours
✓ states the distribution is normal as the sample size is large ✓ gives the value of the mean ✓ gives the value of the variance

(c) What is the probability that the sample proportion of businesses that fail by the end of the year is less than 0.18? (2 marks)

Solution
$P(\hat{p} < 0.18) = 0.1318$
Specific behaviours
✓ uses the correct value of mean and standard deviation ✓ obtains the correct probability

(d) By January 2019, 90 of the 500 new businesses had failed. Calculate a 95% confidence interval for the proportion of new businesses that fail in the first year. (2 marks)

Solution
Sample proportion $\hat{p} = \frac{90}{500} = 0.18$ 95% confidence interval $\left(0.18 - 1.96 \times \sqrt{\frac{0.18 \times 0.82}{500}},\ 0.18 + 1.96 \times \sqrt{\frac{0.18 \times 0.82}{500}}\right)$ $= (0.1463, 0.2136)$
Specific behaviours
✓ calculates the lower bound of the interval correctly ✓ calculates the upper bound of the interval correctly

The business advisory group believes that the proportion of new businesses that fail within a year can be reduced by providing financial advice. They took another random sample of 500 businesses that started in January 2019 and provided them with regular financial advice. In this random sample, at the end of the year 80 businesses had failed.

(e) Calculate the sample proportion and its margin of error at the 95% confidence level. (2 marks)

Solution
Sample proportion $\hat{p} = \frac{80}{500} = 0.16$ $E = 1.96 \times \sqrt{\frac{0.16 \times 0.84}{500}} = 0.0321$
Specific behaviours
✓ calculates the sample proportion correctly ✓ calculates E correctly

(f) Calculate a 95% confidence interval for the proportion of businesses that failed. What do you conclude regarding the value of the financial advice provided to the new businesses? (4 marks)

Solution
Sample proportion $\hat{p} = \frac{80}{500} = 0.16$ 95% confidence interval $\left(0.16 - 1.96 \times \sqrt{\frac{0.16 \times 0.84}{500}},\ 0.16 + 1.96 \times \sqrt{\frac{0.16 \times 0.84}{500}}\right)$ $= (0.1279, 0.1921)$ Comparing this confidence interval with the previous one $(0.1463, 0.2136)$, we can see that they overlap. Therefore, it does not appear that the financial advice has reduced the proportion of businesses that fail in the first year.
Specific behaviours
✓ calculates the lower bound of the interval correctly ✓ calculates the upper bound of the interval correctly ✓ conclusion refers to the confidence intervals overlapping ✓ states the correct conclusion

(g) If the sample size was reduced, what would be the effect on the confidence interval? Justify your answer. (2 marks)

Solution
The width of the confidence interval would be increased, as the margin of error of the sample proportion will increase, thus increasing the error.
Specific behaviours
✓ claims the width of the interval would increase ✓ refers to the increase in margin of error **or** increase in error

Question 12 (continued)

(h) State **two** assumptions that the analyst made in recommending the use of the binomial model in this case and discuss whether they are valid. (4 marks)

Solution
1. We assume that the businesses fail independently of each other. This is unlikely to be valid because: (a) two similar businesses may both fail or both survive (b) if two similar businesses in an area then one may dominate and the other fails. 2. We assume that the probability of a business failing is the same for all businesses. This is unlikely to be valid, as businesses of different types are expected to have different probabilities of failing. 3. The probability of failure is fixed (for the year). This is unlikely to be true, and the probability will depend on changing conditions over time.
Specific behaviours
✓ states one assumption ✓ states with reasons that assumption is unlikely to be valid ✓ states second assumption ✓ states with reasons that assumption is unlikely to be valid

Question 13 **(7 marks)**

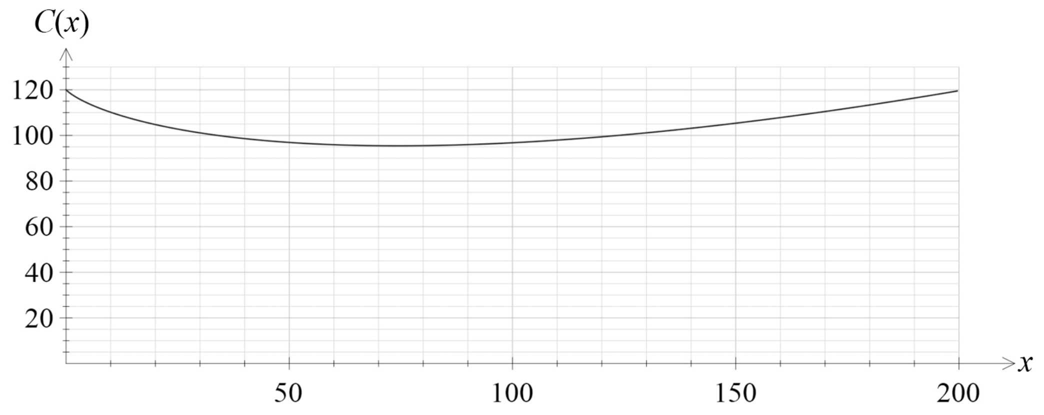
A company manufactures small machine components. They can manufacture up to 200 of a particular component in one day. The total cost, in hundreds of dollars, incurred in manufacturing the components is given by: $C(x)=\dfrac{x\ln(2x+1)}{3}-2x+120$, where x is the number of components that will be produced on that day.

(a) Determine the total cost of manufacturing 20 components in one day. (1 mark)

Solution
$C(20)=\dfrac{20\ln(41)}{3}-40+120=104.7571$ i.e. $\$10475.71\approx\$10476$
Specific behaviours
✓ determines the correct cost

(b) On the axes below, sketch the graph of $C(x)$. (3 marks)

Solution



Specific behaviours
✓ graph covers correct domain
✓ $C(0)$ and $C(200)$ are correct
✓ minimum between 70 and 80

(c) With reference to your graph in part (b), explain how many components the company should manufacture per day if the total cost is to be as low as possible. (3 marks)

Solution
The minimum is at $x=74.205$ $C(74)=95.4307$ i.e. $\$9543.07$ $C(75)=95.4320$ i.e. $\$9543.20$ The company should manufacture 74 components
Specific behaviours
✓ states graph is a minimum at x =74.205
✓ determines cost values for x =74 and x =75
✓ states that 74 components should be manufactured per day

Question 14 **(10 marks)**

A suburban council hires a consultant to estimate the proportion of residents of the suburb who use its library.

(a) The consultant decides to estimate a 95% confidence interval for the proportion to within an error of 0.01. What minimum sample size should be selected? (3 marks)

Solution
$n > \left(\frac{1.96\sqrt{0.5 \times 0.5}}{0.01}\right)^2 = 9604$
Specific behaviours
✓ uses the correct z-value ✓ uses 0.5 in the expression for standard error ✓ determines the sample size (as an integer)

(b) If resource limitations dictate that the maximum sample size that can be managed is 500, what is the maximum margin of error in estimating a 99% confidence interval? (3 marks)

Solution
$\varepsilon = 2.5758 \times \sqrt{\frac{0.5 \times 0.5}{500}} = 0.058$ that is, within 5.8%
Specific behaviours
✓ uses the correct z-value ✓ uses 0.5 in the expression for standard error ✓ calculates the error

The consultant decides to select the sample by standing on the roadside outside the library at lunchtime and asking a random sample of the passers-by whether they use the library.

(c) Identify and explain **two** possible sources of bias with this sampling scheme. (4 marks)

Solution
1. The sample is at a fixed time, so only people around at that time will be sampled. 2. The location is fixed, so: (i) only people at that location will be sampled **or** (ii) not everyone from the suburb will pass by that area, so this is not a random sample of the residents.
Specific behaviours
✓ identifies one possible source of bias ✓ explains why it is a possible source of bias ✓ identifies another possible source of bias ✓ explains why it is a possible source of bias

Question 15 **(9 marks)**

A chef needs to use an oven to boil 100 mL of water in five minutes for a new experimental recipe. The temperature of the water must reach 100 °C in order to boil. The temperature, T, of 100 mL of water t minutes after being placed in an oven set to T_0 °C can be modelled by the equation below.

$$T(t) = T_0 - 175e^{-0.07t}$$

In a preliminary experiment, the chef placed a 100 mL bowl of water into an oven that had been heated to $T_0 = 200$ °C.

(a) What is the temperature of the water at the moment it is placed into the oven? (1 mark)

Solution
$T(0) = 200 - 175e^{-0.07(0)}$ $= 25$ °C
Specific behaviours
✓ states correct temperature

(b) What is the temperature of the water five minutes after being placed in the oven? (1 mark)

Solution
$T(5) = 200 - 175e^{-0.07(5)}$ $= 76.68$ °C
Specific behaviours
✓ states correct temperature

(c) What change could be made to the temperature at which the oven is set in order to achieve the five-minute boiling requirement? (2 marks)

Solution
$100 = T_0 - 175e^{-0.07(5)}$ $T_0 = 100 + 175e^{-0.07(5)}$ ≈ 223 °C
Specific behaviours
✓ states correct equation to be solved ✓ solves for T_0, giving changed temperature

Question 15 (continued)

Assume that T_0 is still $200\ °C$.

(d) Determine the rate of increase in temperature of the water five minutes after being placed in the oven. Give your answer rounded to two decimal places. (2 marks)

Solution
$T'(t) = 12.25e^{-0.07t}$ $T'(5) = 12.25e^{-0.07(5)}$ $= 8.63\ °C/min$
Specific behaviours
✓ states correct derivative of T with respect to t ✓ calculates correct rate

(e) Explain what happens to the rate of change in the temperature of the water as time increases and how this relates to the temperature of the water. (3 marks)

Solution
As time increases, the rate of change in the temperature of the water $\to$ 0. The temperature of the water $\to$ the constant value of T_0.
Specific behaviours
✓ states that the rate of change in the temperature $\to$ 0 ✓ states the water temperature approaches a constant ✓ states the water temperature approaches T_0

Question 16 **(7 marks)**

A large refrigerator in a scientific laboratory is always required to maintain a temperature between 0 °C and 1 °C to preserve the integrity of biological samples stored inside. A scientist working in the laboratory suspects that the refrigerator is not maintaining the required temperature and decides to record the temperature every hour for seven days. Based on these measurements, the scientist concludes that the temperature, T, in the refrigerator is normally distributed with a mean of 0.8 °C and a standard deviation of 0.4 °C.

(a) Temperature in degrees Fahrenheit, T_f, is given by $T_f = \frac{9}{5}T + 32$. Determine the mean and standard deviation of the refrigerator temperature in degrees Fahrenheit. (2 marks)

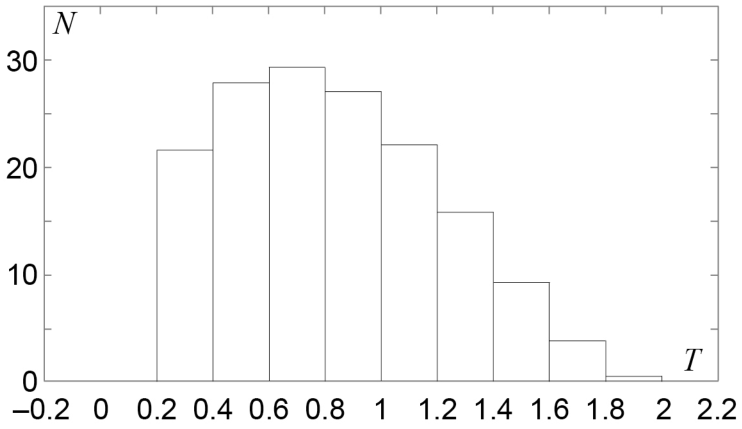
Solution
The mean of T_f is $\mu_{T_f} = \frac{9}{5}\mu_T + 32$ $= \frac{9}{5}\left(\frac{4}{5}\right) + 32$ $= \frac{836}{25} = 33\frac{11}{25} = 33.44$ The standard deviation of T_f is $\sigma_{T_f} = \frac{9}{5}\sigma_T$ $= \frac{9}{5}\left(\frac{2}{5}\right)$ $= \frac{18}{25} = 0.72$
Specific behaviours
✓ determines correct mean ✓ determines correct standard deviation

(b) Determine the probability that the refrigerator temperature is above 1 °C. Give your answer rounded to four decimal places. (1 mark)

Solution
$P(T > 1) = 0.3085$
Specific behaviours
✓ determines correct probability

Question 16 (continued)

The histogram of data gathered by the scientist is shown below. N denotes the number of observations in each temperature interval.



(c) Do you agree that the normal distribution was an appropriate model to use? Provide a reason to justify your response. (2 marks)

Solution
No. The distribution appears to be skewed to the right (non-symmetric)
Specific behaviours
✓ recognises that the normal distribution was not an appropriate model ✓ justifies conclusion based on lack of symmetry of histogram

An alternative probability density function proposed to model the refrigerator temperature in degrees Celcius, is given by

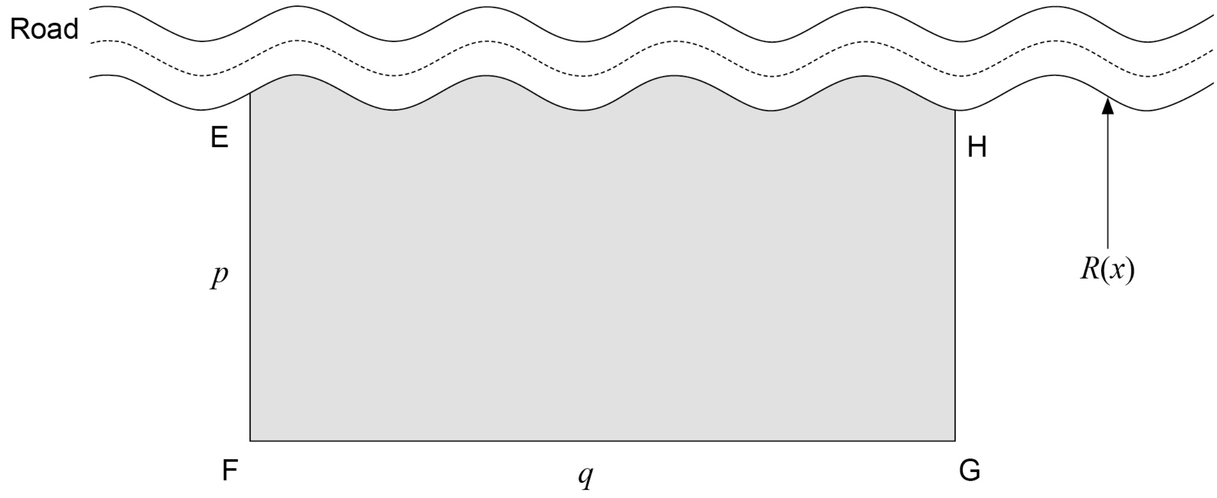
$$p(t) = \frac{3}{4}t^3 - 3t^2 + 3t\,,\ \ 0 \le t \le 2$$

(d) Determine the probability that the refrigerator temperature is above $1\ °\mathrm{C}$ using the new model. (2 marks)

Solution
$P(T \ge 1) = \int_1^2 p(t)\,dt$ $= \int_1^2 \frac{3}{4}t^3 - 3t^2 + 3t\,dt$ $= \left[\frac{3}{16}t^4 - t^3 + \frac{3}{2}t^2\right]_1^2$ $= 1 - \frac{11}{16}$ $= \frac{5}{16}$ {0.3125}
Specific behaviours
✓ identifies integral to determine probability ✓ determines correct probability

Question 17 **(12 marks)**

David and Katrina have a small farm and wish to fence off an area of their land so they can raise sheep. The area they have chosen has one border along a road as shown in the diagram below.



The enclosure is shown as the shaded area above and has right angles at points F and G. David and Katrina want the combined lengths of the fencing from E to F and F to G to equal 500 metres. Let the length of fence EF be equal to p metres and the length of fence FG be equal to q metres. If we locate the origin at the point F and the x-axis along the line FG, the equation defining the fence along the road is given by:

$$R(x) = 10\sin\left(\frac{x}{15}\right) + p$$

(a) Show that the equation defining the area of the enclosure, $A(q)$, can be given in terms of q as follows:

$$A(q) = 500q - 150\cos\left(\frac{q}{15}\right) - q^2 + 150$$ (4 marks)

Solution
$A(p,q) = \int_0^q \left(10\sin\left(\frac{x}{15}\right) + p\right) dx$ $= pq - 150\cos\left(\frac{q}{15}\right) + 150$ $p + q = 500$ $\therefore p = 500 - q$ $A(q) = q(500 - q) - 150\cos\left(\frac{q}{15}\right) + 150$ $= 500q - 150\cos\left(\frac{q}{15}\right) - q^2 + 150$
Specific behaviours
✓ states correct integral for area ✓ evaluates integral to determine equation for area in terms of p and q ✓ states that $p + q = 500$ ✓ substitutes for p to obtain the required result

Question 17 (continued)

(b) Determine, to the nearest metre, the value of q that will allow the sheep to graze over the maximum area and state this maximum area. (4 marks)

Solution
$A'(q) = 500 + 10\sin\left(\frac{q}{15}\right) - 2q$ $0 = 500 + 10\sin\left(\frac{q}{15}\right) - 2q$ $q \approx 247$ $A''(q) = \frac{2}{3}\cos\left(\frac{q}{15}\right) - 2$ $A''(247) = -ve\ \{-2.48\}$ $\therefore$ maximum $A(247) = 62750$ $\therefore$ maximum area = 62750 m^2
Specific behaviours
✓ differentiates the area equation   ✓ sets the derivative to 0 and solves it to obtain q ✓ obtains the second derivative (or draws a sign diagram for the derivative) to conclude that the point is a global maximum   ✓ states the maximum area

The length of the fence from E to H is given by the equation:

$L_{EH} = \int_0^q \sqrt{1 + \left(R'(x)\right)^2}\, dx$, where $R'(x)$ is the first derivative of $R(x)$.

(c) (i) Determine $R'(x)$. (1 mark)

Solution
$R'(x) = \frac{2}{3}\cos\left(\frac{x}{15}\right)$
Specific behaviours
✓ correctly determines $R'(x)$

(ii) Hence determine the total length of fencing required by David and Katrina to enclose their sheep with maximum area for grazing. (3 marks)

Solution
$L_{EH} = \int_0^{247} \sqrt{1+\left(\frac{2}{3}\cos\left(\frac{x}{15}\right)\right)^2}\,dx$ $= 273$ metres $\{272.86\}$ $p = 500 - 247 = 253$ $R(247) = 10\sin\left(\frac{247}{15}\right) + 253$ ≈ 247 Total length of fencing $\approx 253+247+273+247$ ≈ 1020 metres
Specific behaviours
✓ calculates L_{EH} ✓ calculates $R(247)$ ✓ states total length of fencing

An *Acknowledgements variation* document is available on the Authority website.

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